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ON DETECTING NEW BETTER THAN USED PROPERTY OF LIFE DISTRIBUTIONS

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ABSTRACT

A test procedure is proposed for testing the null hypothesis that two life distributions are equal against the alternative that one is more new better than used than the other. The asymptotic normality of the proposed class of test statistics is established. The performance of the test procedure is evaluated by finding Pitman asymptotic relative efficiencies(ARE) for different distributions in comparison with the test procedure due to Hollander ,Park and Proschan[6].

Key Words and Phrases: *NBU, Pitman ARE, U-statistic, Two-sample test.*

INTRODUCTION

Statistical theory and its applications have played a significant role in advancement of methodology and techniques for solving various problems in reliability and survival analysis. The exponential distribution is a popular model which is useful wherever the 'no aging' phenomenon is evident. 'No aging' means that the probability distribution of the life time of a unit does not change with the knowledge that the unit has already survived for a given time. As against this many units exhibit the 'positive aging' phenomena. The term 'positive aging' is used to denote the situation where the performance of a unit deteriorates with its age. Classes of life distributions based on notion of aging have been introduced in the literature. Some of the classes of life distributions based on aging are Increasing failure rate (IFR), Increasing failure rate average (IFRA), New better than used (NBU). The chain of implication of these notions is given by IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU. In practical situations it has been noticed that life time of units posses one or the other aging property. Hence it is of interest to the statisticians to propose tests for the null hypothesis of exponentiality against the alternative of positive aging. In this paper, we consider one of the important classes of life distribution, namely, new better than used (NBU) which is defined below.

Definition 1.1: A life distribution with distribution function F is said to be new better than used (NBU) if

$$\overline{F}(x+y) \leq \overline{F}(x)\,\overline{F}(y), \qquad x, y \geq 0.$$

The new better than used of distributions is important in replacement policies (Barlow and Proschan ,[4]).

Many tests have been proposed for testing exponentiality against the alternative of NBU by researchers. Testing exponentiality against the new better than used alternatives are discussed by Hollander and Proschan [7], Koul [8], Kumazawa [9] and Ahmad ([1], [2]) among others. However, much attention is not given to the problem of testing whether one distribution possess more 'NBU-ness' property than the other. The only test procedure available for such a problem is due to Hollander, Park and Proschan [6]. In this paper, we consider a class of test statistics for the problem of testing exponentiality against new better than used class of alternatives. The test considered is extended to test the hypothesis that the two life distributions are identical against the alternative that one possesses more 'new better than used' property than the other.

The following situation illustrates how a test for the hypothesis that 'the two life distributions are identical against the alternative that one possesses more 'new better than used' property than the other' might prove useful.

A police department is considering buying a fleet of used cars for its detectives. The department is trying to choose between a fleet of Camaros and a fleet of Firebirds of comparable ages which are being offered for sale by a rental cam company. It is known that the mean life times of the two types of cars are equal. In this care the department would prefer the Firebird fleet if it were less NBU than the Camaro fleet. The primary information that the means are equal may be known a prior, or could be checked by application of a standard two-sample test for location differences.

The only test procedure available to the best of our knowledge for the above stated two sample problem is the test procedure due to Hollander, Park and Proschan[6] which is described below.

Hollandar, Park and Proschan [6] proposed a test statistic based on the measure

$$\gamma(F,G) = \gamma(F) - \gamma(G),$$

where $\gamma(F) = \int_0^\infty\int_0^\infty \overline{F}(x+y)dF(x)dFxy$ and $\gamma(G) = \int_0^\infty\int_0^\infty \overline{G}(x+y)dG(x)dGxy$.

Based on two samples $X_1,\ldots,X_k$ and $Y_1,Y_2,\ldots,Y_n$, the statistic proposed is given by

$$V_{k,n} = J_k(x) - J_n(y)$$

where

$$J_k(x) = 2\left[k(k-1)(k-2)\right]^{-1} \sum \psi\,(X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_3}),$$

$\psi(a,b) = 1$ if a>b, 0 if a≤b, and Σ is the sum taken over all m(m-1)(m-2)/2 triples $(\alpha_1, \alpha_2, \alpha_3)$ of three integers between 1 and m inclusive, $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq \alpha_3$ and $\alpha_2 < \alpha_3$, Hollander and Proschan [7] used $J_k(X)$ as a test statistic for the one sample testing problem of exponentiality against (nonexponential) NBU alternatives $J_n(Y)$ is defined analogously, based on $Y_1,\ldots,Y_n$.

Observe that under H_0, $\Delta(F)$ is equal to $\Delta(G)$ and under H_A, $\Delta(F)-\Delta(G)$ is strictly less than 0. Therefore significantly small values of $T_{m,n}$ may indicate that F is 'more NBU' than is G. Similarly significantly large values of $V_{k,n}$ may indicate that F is 'less NBU' than is G. Our test is to reject H_0 in favour of H_A if $V_{k,n}$ is too small.

In section 2, we propose a class of test statistics and studied their asymptotic distributions. The section 3 is devoted to asymptotic relative efficiency comparisons. In section 4, we present the remarks and conclusions.

2. THE PROPOSED TWO-SAMPLE MORE NBU TEST

Let $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_n$ denote two random samples from continuous life distributions F and G, respectively. We want to develop test statistic for testing the null hypothesis

H_0: F = G (the common distribution is not specified)

versus the alternative hypothesis

H_1: F is 'more NBU' than G.

Consider the parameter, for an integer m>1

$$\gamma(F,G) = \gamma(G) - \gamma(F) ,$$

where $\gamma(F) = \int_0^\infty \overline{F}^c(mx)dF(x)$ and $\gamma(G) = \int_0^\infty \overline{G}^c(my)dG(y)$.

Here, $\gamma(F)$ *and* $\gamma(G)$ can be considered as the measure of degree of the NBU-ness. Ahmed [2] test used this measure as basis for their test statistic. If F (G) belongs to NBU, then $\gamma(F) < 0$ $(\gamma(G) < 0)$ and $\gamma(F,G)$ can be taken as a measure by which F is 'more NBU' than G. Under H_0, $\gamma(F,G) = 0$ and it is strictly greater than zero under H_1.

An unbiased estimator for $\gamma(F,G)$, which is defined as

$$T_{k,n}^m = T_n - T_k ,$$

where T_k and T_n are U-statistics with kernels of degree (c+1) which are defined as

$$h_1(x_1, x_2, \ldots, x_{c+1}) = \frac{1}{c+1}\sum_{i=1}^{c+1} I(Min(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{c+1}) > mx_i)$$ and

$$h_2(y_1, y_2, \ldots, y_{c+1}) = \frac{1}{c+1}\sum_{j=1}^{c+1} I(Min(y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{c+1}) > my_j),$$ respectively.

Here *I(.)* is an indicator function.

Asymptotic normality of the test $T_{k,n}^m$

In this subsection, we study the asymptotic distribution of $T_{k,n}^m$. For that define

$$\xi_1(F) = E[\psi_1(X_1)]^2 - [\gamma(F)]^2,$$

where $\psi_1(x_1) = E\{h_1(x, X_2, \ldots, X_{c+1})\}$.

Next, $\xi_1(G)$ is defined as

$$\xi_1(G) = E[\psi_1^*(Y_1)]^2 - [\gamma(G)]^2$$

where

$\psi_1^*(y_1) = E\{h_2(y, Y_2, ..., Y_{c+1})\}$.

The asymptotic normality of the test $T_{k.n}$ is presented in the following theorem.

Theorem 1: The asymptotic distribution of $\sqrt{N}\left[T_{k,n}^m - \gamma(F,G)\right]$ is normal with mean zero and variance given by $\sigma^2(T_{k,n}) = \sigma_1^2 + \sigma_2^2$, where $\sigma_1^2 = \dfrac{(c+1)^2 \xi_1(F)}{\lambda}$ and $\sigma_2^2 = \dfrac{(c+1)^2 \xi_1(G)}{1-\lambda}$, where $\xi_1(F)$ and $\xi_1(G)$ is as defined above.

Under H_0: $F=G=F_0$, then $\xi_1(\mathrm{F}) = \xi_1(\mathrm{G}) = \xi_1(\mathrm{F}_0)$ and if F_0 is exponential distribution function then,

$$(c+1)^2 \xi_1(F_0) = \frac{1}{2mc+1} + \frac{2c}{m(c-1)+1}\left[\frac{1}{mc+1} - \frac{m}{mc(m+1)+1}\right] - \frac{(c+1)^2}{(mc+1)^2} + \frac{c^2}{(m(c-1)+1)^2}\left[\frac{m}{m(2c-1)+2} - \frac{2m}{mc+1}\right] + \frac{c^2}{(m(c-1)+1)^2}$$

The approximate α-level test rejects H_0 in favour of H_1, if $\dfrac{\sqrt{N}\, T_{k,n}^m}{\sigma^2(T_{k,n})} > z_\alpha$, where Z_α is the upper α - percentile point of standard normal distribution. Since, $\gamma(F,G) > 0$ under H_1 and from the asymptotic normality of $T_{k,n}^m$, the test based on $T_{k,n}^m$ is consistent against the alternative F is 'more NBU than' G.

3. ASYMPTOTIC RELATIVE EFFICIENCY

We study the asymptotic relative efficiency of $T_{k,n}^m$, relative to the $\mathrm{V}_{\mathrm{k,n}}$ test of Hollander, Park and Proschan[6]. The asymptotic efficacies of the test due to Hollander, Park and Proschan[6] for various alternatives are computed and presented in sub section 3.1. The asymptotic efficacies of the newly proposed test and asymptotic relative efficiencies are studied in subsection 3.2.

3.1. Asymptotic Efficacies of the test due to Hollander et.al.,[6]

The Pitman efficacies of two sample test based on Hollander, Park and Proschan [6] is determined by specifying a common distribution with parameter θ in the null hypothesis and by considering sequence of alternatives $(F_{\theta\phi_N}, F_\theta)$ where $\phi_N = 1 + \dfrac{a}{\sqrt{N}}, a > 0$ be constant.

The Pitman asymptotic efficacy is

$$Eff(V_{k,n}, F_{i,\theta}, G) = \sigma_0^{-2}(V_{k,n})\left\{\frac{d}{d\theta}\gamma(F_{i,\theta}, G)\right\}_{\theta \to \theta_0}$$

The sub-sequence of distribution considered here are $(F_{1,\theta\phi_N}, F_{1,\theta})$, $(F_{2,\theta\phi_N}, F_{2,\theta})$ and $(F_{3,\theta\phi_N}, F_{3,\theta})$ where $F_{1,\theta}$, $F_{2,\theta}$ and $F_{3,\theta}$ are defined below:

1. Weibull distribution,

$\overline{F}_{1,\theta\phi}(x) = \exp(-x^{\theta\phi}), \quad x > 0, \quad \theta \geq 1$ and

$\overline{F}_{1,\theta}(x) = \exp(-x^{\theta}), \quad x > 0, \quad \theta \geq 1.$

2. Makeham Distirbution

$\overline{F}_{2,\theta\phi}(x) = \exp[-x + \theta\phi(x + e^{-x} - 1)], \quad x > 0, \quad \theta \geq 0$ and

$\overline{F}_{2,\theta}(x) = \exp[-x + \theta(x + e^{-x} - 1)], \quad x > 0, \quad \theta \geq 0.$

3. Gamma Distribution

$\overline{F}_{3,\theta\phi}(x) = 1 - \int_0^x \frac{1}{\Gamma(\theta\phi)} u^{\theta\phi - 1} e^{-u} du, \quad x > 0, \quad \theta \geq 0$ and

$\overline{F}_{3,\theta}(x) = 1 - \int_0^x \frac{1}{\Gamma(\theta)} u^{\theta - 1} e^{-u} du, \quad x > 0, \quad \theta \geq 0.$

The Pitman efficacies of Hollander, Park and P`roschan [6] are given in table 1 below.

TABLE 1

θ	$(F_{1,\theta\phi_N}, F_{1,\theta})$	$(F_{2,\theta\phi_N}, F_{2,\theta})$	$(F_{3,\theta\phi_N}, F_{3,\theta})$
2	2.586	0.2006	0.3824
3	0.711	0.2582	2.9812
4	2.061	0.3036	3.8832
5	1.057	0.3409	3.5395
6	1.653	0.3725	2.5771
7	1.667	0.3996	1.5776
8	1.383	0.4235	0.8410
9	1.130	0.4446	0.4006
10	1.178	0.4636	0.1737

3.2. Asymptotic Efficiency of the test based on $T^{m}_{k,n}$

We study the asymptotic relative efficiency of $T^{m}_{k,n}$, relative to the $V_{k,n}$ test of Hollander, Park and Proschan[6] for the two pairs of distributions $(F_{i,\theta}, G)$. Here, we assume that G is an exponential distribution with mean one. We consider $F_{1,\theta}$ as Weibull distribution and $F_{2,\theta}$ as Linear failure rate distribution as defined below:

1. Weibull Distribution:

$\overline{F}_{1,\theta}(x) = \exp\{-x^{\theta}\}$, θ>0, x ≥θ .

2. Linear Failure Rate Distribution

$$\overline{F}_{2,\theta}(x) = \exp\left[-(x+\theta\frac{x^2}{2})\right], \quad x>0, \quad \theta \geq 0.$$

The ARE's of the proposed test $T_{k,n}^{m}$ with respect to the test $V_{k,n}$ of Hollander, Park and Proschan(1986) for Weibull distribution with parameter θ, $F_{1,\theta}$ and Linear failure rate distribution are respectively 1.8432 and 2.6923.

Next, we compute the asymptotic relative efficiency of the two sample test based on $T_{k,n}^{m}$ relative to the test due to Hollander, Park and Proschan [6], by specifying the common null distribution in the null hypothesis as F_θ with θ≥1 and considering sequence of alternatives $(F_{\theta\phi_N}, F_\theta)$, where $\phi = 1+\frac{a}{\sqrt{N}}$, a being arbitrary positive constant. Note that as $N \to \infty$, the sequence of alternatives converges to the null hypothesis. The efficacy of the $T_{k,n}^{m}$ test is given by

$$Eff(T_{k,n}^{m}) = \frac{\left[\gamma'(F_{\theta\phi_N}, F_\theta)\right]^2}{\sigma_0^2(T_{k,n}^{m})},$$

where $\sigma_0^2(T_{k,n}^{m})$ is null asymptotic variance of $\sqrt{N}\ T_{k,n}^{m}$, and

$$\gamma'(F,G) = \left[\frac{d\gamma(F_{\theta\phi_N}, F_\theta)}{d\phi}\right]_{\phi=1}.$$

The sequence of alternatives considered here are $(F_{1,\theta\phi_N}, F_{1,\theta})$, $(F_{2,\theta\phi_N}, F_{2,\theta})$ and $(F_{3,\theta\phi_N}, F_{3,\theta})$ whose functional forms are given as in subsection 3.1.

The Asymptotic relative efficiencies of the proposed test $T_{k,n}^{m}$ relative to the test due to Hollander, Park and Proschan[6] $V_{k,n}$ for the various alternatives are presented in table 2, 3 and 4 respectively.

TABLE 2 ARE of $T_{k,n}^{m}$ w.r.t. $V_{k,n}$ for $(F_{1,\theta\phi_N}, F_{1,\theta})$

m → θ	2	3	4	5
2	1.3522	1.2725	1.1414	1.0246
3	4.7063	3.6523	2.8459	2.2945
4	1.432	0.9118	0.6191	0.4482
5	2.3486	1.2304	0.7358	0.4699
6	1.2243	0.5354	0.2804	0.1574
7	1.3871	0.5089	0.2463	0.1144
8	0.9226	0.2865	0.1247	0.0485
9	0.6982	0.2443	0.0912	0.0294
10	0.6258	0.1573	0.0519	0.01348

TABLE 3 ARE of $T_{k,n}^{m}$ w.r.t. $V_{k,n}$ for $(F_{2,\theta\phi_N}, F_{2,\theta})$

m → θ	2	3	4	5
2	2.8221	2.7510	2.6841	2.6285
3	2.7694	2.6801	2.6071	2.5467
4	2.7515	2.6501	2.5712	2.5081
5	2.7497	2.6385	2.5548	2.4889
6	2.7558	2.6365	2.5486	2.4804
7	2.7663	2.6398	2.5483	2.4779
8	2.7795	2.6467	2.5519	2.4796
9	2.7940	2.6555	2.5576	2.4835
10	2.8097	2.6660	2.5653	2.4895

TABLE 4 ARE of $T_{k,n}^{m}$ w.r.t. $V_{k,n}$ for $(F_{3,\theta\phi_N}, F_{3,\theta})$

m → θ	2	3	4	5
2	2.5776	2.8234	2.8750	2.8412
3	1.0864	0.8754	0.6946	0.5582
4	1.4023	0.9043	0.5902	0.4003
5	1.8609	1.0739	0.5855	0.3397
6	3.3181	1.4374	0.6602	0.3295
7	6.0033	2.1549	0.8368	0.3608
8	11.9535	3.5692	1.1751	0.4379
9	26.0950	5.4312	1.2905	0.3606
10	6.4387	1.2801	0.2912	2.4733

4. SOME REMARKS:

1. The Asymptotic relative efficiencies of the proposed test with respect to the test due to Hollander, Park and Proschan [6] is computed for two pairs of distributions (F_θ,G) with G is exponential with mean one and F_θ as Weibull and Linear failure rate distributions.
2. It is observed that the proposed test performs better for the alternatives considered F_θ is either Weibull or Linear failure rate distributions when G is exponential.
3. The asymptotic relative efficiencies of the test proposed with respect to Hollander, Park and Proschan[6] is computed for three pairs of distributions $(F_{1,\theta\phi_N}, F_{1,\theta})$, $(F_{2,\theta\phi_N}, F_{2,\theta})$ and $(F_{3,\theta\phi_N}, F_{3,\theta})$ with F_1, F_2, F_3 as Weibull, Makeham and Gamma distributions respectively.
4. It is observed that the proposed test performs better than the test due to Hollander, Park and Proschan[6] when the underlying distribution is Makeham or Gamma.
5. The test due to Hollander, Park and Proschan[6] performs better for Weibull and Linear failure rate distributions.
6. The optimum value of m to use the proposed two sample test is 2.
7. Hence, if the data under consideration is exactly NBU , the new test proposed would be a better choice.

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